

B.I.E.M. in Solid Mechanics

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In solid mechanics the weak formulation produces an integral equation ready for a discretization [1,2] and with less restrictive requirements than the standard field equations. Fundamentally the weak formulation is the expression of a Green formula.

An alternative is to choose another Green formula materializing a reciprocity relationship between the basic unknowns and an auxiliary family of functions. The degree of smoothness required to practice the discretization is then translated to the auxiliary functions.

The subsequent discretization (constant, linear, etc [3]) produces a set of equations on the boundary of the domain. For linear 3-D problems the BIEM appears then as a powerful alternative to FEM, because of the reduction to 2-D thanks to the features previously described.

The weak formulation for a general problem can be established from a field equation

$$A u = f \quad (1)$$

where A is a symmetric operator.

Using the weighted residuals philosophy the inner product with an auxiliary function is written

$$(A u, \psi) = (f, \psi) \quad (2)$$

An integration by parts of the L.H.S. produces a bilinear form plus boundary conditions

$$a(u, \psi) + b_1(u, \psi) = (f, \psi) \quad (3)$$

some derivatives are transferred from the field variable u to the auxiliary one ψ , allowing the use of less restrictive conditions for the approximation.

In elasticity this equations is the virtual work principle

$$\int_{\Omega} \epsilon_{ij} \sigma_{ij} d\Omega - \int_{\partial\Omega} \psi_i T_i^v d\Gamma = \int_{\Omega} x_i \psi_i d\Omega \quad (4)$$

A discretization of the kind

$$u \approx u_n = c_i \psi_i \quad i = 1, 2, \dots, n$$

produces the typical F.E.M. formulation.

Another series of integration of (3) leads to the mirror image of (2)

$$(u, A\psi) + b_1(u, \psi) + b_2(u, \psi) = (f, \psi) \quad (5)$$

In elasticity the Maxwell-Betti reciprocity theorem

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$$-\int_{\Omega} u_i \sigma_{ij,j}^* d\Omega + \int_{\partial\Omega} u_i T_i^{v*} d\Gamma - \int_{\partial\Omega} \psi_i T_i^v d\Gamma = \int_{\Omega} x_i \psi_i d\Omega \quad (6)$$

Taking as auxiliary function ψ_i the solution to the Kelvin's problem of a concentrated load at \underline{x}

$$x_i = -\sigma_{ij,j}^* = \Delta(y - \underline{x})$$

equation (6) for a zero body forces problem is transformed into

$$u_j(\underline{x}) + \int_{\partial\Omega} T_{ji}(\underline{x}, \underline{y}) u_i(\underline{y}) d\Gamma(\underline{y}) = \int_{\partial\Omega} U_{ji}(\underline{x}, \underline{y}) t_i(\underline{y}) d\Gamma(\underline{y}) \quad (7)$$

where T_{ij} and U_{ij} are known functions representing tractions and displacements on boundary points \underline{y} when a concentrated unit point load is applied at \underline{x} in the j direction, considering the body as part of an infinite medium.

In (7) \underline{x} is an internal point. In order to produce a boundary equation, \underline{x} is taken to the boundary and (7) changes to

$$C_{ij}(\underline{x}) u_i(\underline{x}) + \int_{\partial\Omega} T_{ji}(\underline{x}, \underline{y}) u_i(\underline{y}) d\Gamma(\underline{y}) = \int_{\partial\Omega} U_{ji}(\underline{x}, \underline{y}) t_i(\underline{y}) d\Gamma(\underline{y}) \quad (8)$$

where for smooth boundaries $C_{ij}(\underline{x}) = \frac{1}{2} \delta_{ij}$ (δ_{ij} : Kronecker function)

The same formulation can be established for elastodynamics with the only difference that equation (6) is written for the Fourier transformed states. The fundamental solution, ψ , will be here the response of an infinite medium to a unit harmonic force concentrated at \underline{x} .

In order to solve the integral equation (8) the boundary is discretized and the values of u_i and t_i interpolated over the boundary elements with the FEM philosophy to obtain a system of equations of the form

$$\underline{G} \underline{t} = \underline{H} \underline{u} \quad \text{where } \underline{t} \text{ and } \underline{u} \text{ are nodal vectors.}$$

Equation (7) and its derivatives will produce the displacement and stress at any internal point.

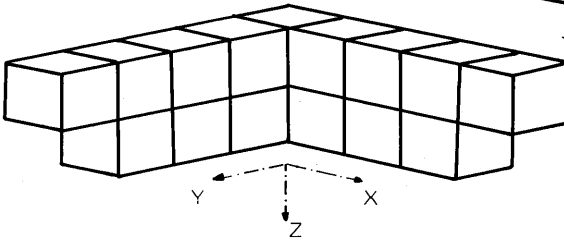
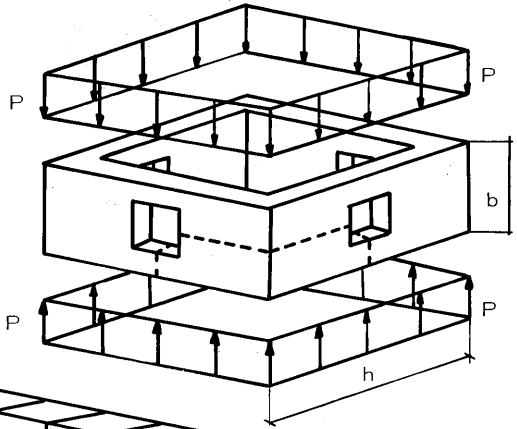
Applications

In the following, two examples are shown. The first corresponds to a elastostatic 3-D problem where the symmetry was taken into account and only one eighth of it was discretized. The tractions and displacements were considered to be constants over the boundary elements. In the second case the dynamic stiffness of a strip footing is obtained assuming the soil to be an isotropic elastic half space with a hysteretical damping of 5%. In theory the surface should be discretized to infinity. In the figure, the variation of the real part of the swaying stiffness is presented versus the amount of discretized free field when one increases the number of boundary elements. Some 3-D problems can be found in [4] and some more will be presented at the conference.

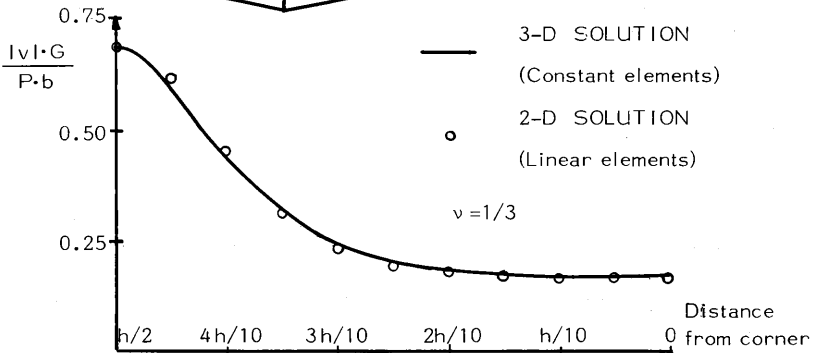
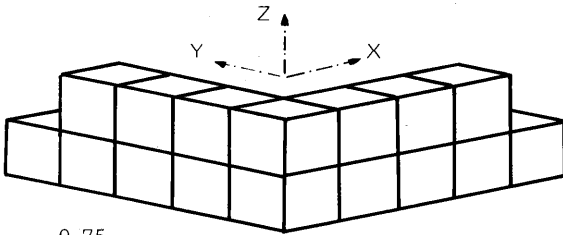
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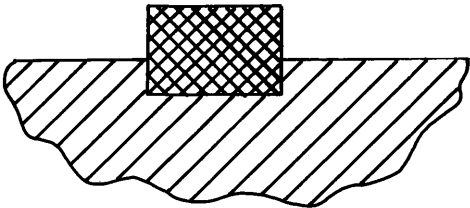
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OPENINGS
UNDER CONSTANT
PRESSURE



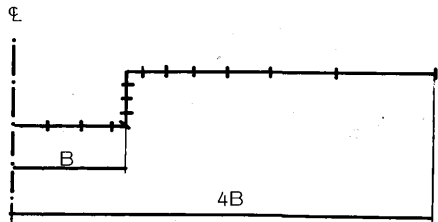
BOUNDARY
DISCRETIZATION



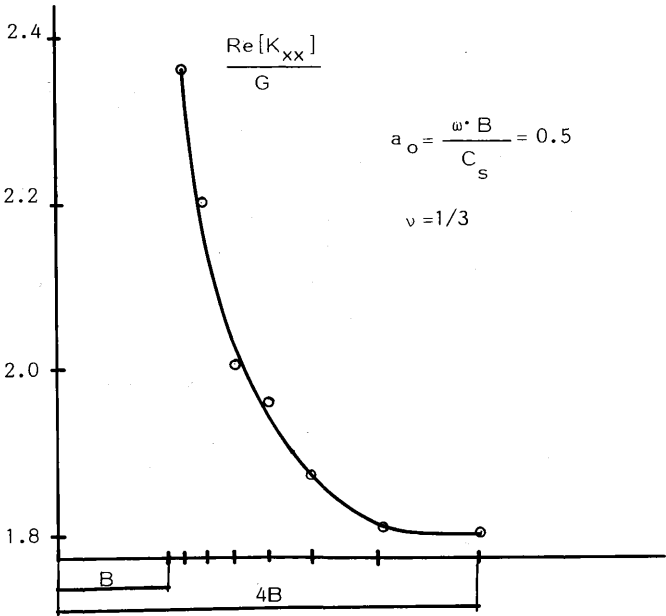
VERTICAL DISPLACEMENTS AT UPPER AND LOWER SURFACES



STRIP FOOTING



BOUNDARY DISCRETIZATION



VARIATION OF THE REAL PART OF THE SWAYING STIFFNESS WITH THE AMOUNT OF FREE FIELD DISCRETIZED